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THE KINEMATICS OF THE ROLLING OF A RIGID BODY ALONG TWO LINES†

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The general differential equations of the kinematics of the rolling of a smooth rigid surface along two smooth rigid lines without sliding are investigated. A formula for the velocity of rotation of this surface relative to the path traversed is obtained and the types of characteristic points of the rolling are specified. Examples are given.

In a previous paper [1], the system of differential equations of the rolling of a smooth rigid surface without sliding along two smooth rigid directrices was derived, in which the length of the arc of one of the directrices was chosen as the independent variable. This not only introduced asymmetry into the basic system of the equations but also made a local nature of the discussion inevitable in cases when, in the kinematical meaning of the problem of rolling, this length varies non-monotonically.

Unlike this approach, an arbitrary parameter is chosen below as the independent variable. An important example of such a parameter is the total angle of rotation of the body when the body rolls. In this connection, a formula is derived for the velocity of this rotation with respect to the path traversed, which, in particular, leads to the specification of the natural types of characteristic points of rolling. This approach enables the above-mentioned drawbacks to be avoided but introduces its own difficulties. Thus, the equations become more complicated when the angle of rotation is chosen, and, moreover, one is forced to go beyond the scope of "purely mathematical" proofs (see [2]). In particular, the possibility of rolling itself is assumed without mathematical proof. It is useful to bear both approaches in mind.

There is an extensive bibliography, beginning with Painlevé's publications (e.g. [3]), on the related problem of taking sliding into account in the dynamics of the rolling of a rigid body along directrices.

1. THE DIFFERENTIAL EQUATIONS OF ROLLING FOR THE CASE OF AN ARBITRARY ARGUMENT

Suppose two directrices are specified in space and are oriented, sufficiently smooth lines L_1 and L_2 without common points and have the vector equations $\mathbf{r}=\mathbf{f}_i(s_i)$ $(i=1, 2)$, where $|s_i|$ is the arc-length of the line L_i measured from a certain point on it, while an increase in s_i corresponds to the orientation of this line. Suppose further that an oriented, sufficiently smooth surface Q is given by the vector equation $\mathbf{r}=\mathbf{g}(u, \upsilon)$, and that the vector $\mathbf{n}(u, \upsilon):=\mathbf{g}_u(u, \upsilon)\times\mathbf{g}_\upsilon(u, \upsilon)/|\mathbf{g}_u(u, \upsilon)\times\mathbf{g}_\upsilon(u, \upsilon)|$ is directed along the outward normal to Q, where Q is the boundary of a certain body.

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We shall assume that rolling of the surface Q along the lines L_1 and L_2 is determined by a family of motions $K^\alpha : Q \mapsto K^\alpha Q$ which depend fairly smoothly on a certain parameter α, and, for each value of α, the surface $K^\alpha Q$ has the single common point M_i^α with the line L_i $(i=1, 2)$ at which L_i touches $K^\alpha Q$. In addition, it is required that near M_i^α the line L_i should be on the exterior side of $K^\alpha Q$ and the point M_i^α should depend continuously on α.

Denote the value of s_i for the point M_i^α by s_i^α and the coordinates of the point $N_i^\alpha := (K^\alpha)^{-1} M_i^\alpha$ by u_i^α, υ_i^α. Then, by increasing α by $d\alpha$ using the condition that there is no sliding, and then changing from differentials to derivatives, we arrive at the following system of five non-linear differential equations:

$$\begin{gathered}
|\mathbf{g}_u(u_i,\upsilon_i)u_i' + \mathbf{g}_\upsilon(u_i,\upsilon_i)\upsilon_i'| = |s_i'| \\
[\mathbf{g}(u_2,\upsilon_2) - \mathbf{g}(u_1,\upsilon_1)]\cdot[\mathbf{g}_u(u_i,\upsilon_i)u_i' + \mathbf{g}_\upsilon(u_i,\upsilon_i)\upsilon_1'] = \\
= [\mathbf{f}_2(s_2) - \mathbf{f}_1(s_1)]\cdot \mathbf{f}_i'(s_i)s_i' \quad (i=1,2) \\
[\mathbf{g}_u(u_1,\upsilon_1)u_1' + \mathbf{g}_\upsilon(u_1,\upsilon_1)\upsilon_1']\cdot[\mathbf{g}_u(u_2,\upsilon_2)u_2' + \mathbf{g}_\upsilon(u_2,\upsilon_2)\upsilon_2'] = \mathbf{f}_1'(s_1)\cdot \mathbf{f}_2'(s_2)s_1's_2'
\end{gathered} \tag{1.1}$$

where the prime on u_i', υ_i', s_i' denotes the derivative with respect to α and the superscript α is omitted for brevity. In addition, the following equality must hold

$$\begin{gathered}
[\mathbf{g}(u_2,\upsilon_2) - \mathbf{g}(u_1,\upsilon_1)]\cdot\{[\mathbf{g}_u(u_1,\upsilon_1)u_1' + \mathbf{g}_\upsilon(u_1,\upsilon_1)\upsilon_1']\times[\mathbf{g}_u(u_2,\upsilon_2)u_2' + \mathbf{g}_\upsilon(u_2,\upsilon_2)\upsilon_2']\} = \\
= [\mathbf{f}_2(s_2) - \mathbf{f}_1(s_1)]\cdot[\mathbf{f}_1'(s_1)\times \mathbf{f}_2'(s_2)]s_1's_2'
\end{gathered}$$

and it is essentially that only the signs of the left-hand and right-hand sides should be the same since the coincidence of their moduli follows from Eqs (1.1).

The six required functions s_i, u_i, υ_i $(i=1, 2)$ of α are also related by two finite equations. One of them is obvious, namely

$$|\mathbf{g}(u_2,\upsilon_2) - \mathbf{g}(u_1,\upsilon_1)| = |\mathbf{f}_2(s_2) - \mathbf{f}_1(s_1)| \tag{1.2}$$

The other one has a more complicated form and can be obtained by eliminating u_i', υ_i', s_i' from system (1.1) (see [1]) or from geometrical considerations [4]. Taking these finite equations into account we obtain that the general solution of system (1.1) includes three arbitrary constants once the parameter α has been eliminated from it. Hence, the set of geometrically different motions is, in general, three-dimensional.

It is essential that the system of equations (1.1) should be homogeneous with respect to u_i', υ_i', s_i'. It therefore only specifies the trajectories on the four-dimensional manifold defined by the above finite equations but not the law of motion along them as a function of α. For a more precise definition of this law it is required to indicate the specific meaning of the parameter α; thus, we may assume $\alpha = s_1$ [1]. Other examples will be given in Section 4.

2. THE VELOCITY OF ROTATION WHEN THERE IS ROLLING

Theorem. Suppose that, during rolling, the surface $K^\alpha Q$ has rotated through an angle $d\psi$ about the axis $M_1^\alpha M_2^\alpha$ in the positive direction. We then have

$$d\psi = \left[\frac{\mathbf{c}_i^\alpha \cdot \mathbf{f}_i''(s_i^\alpha)}{|\mathbf{c}_i^\alpha|^2} - \frac{\operatorname{sgn}(\boldsymbol{\nu}_i^\alpha \cdot \mathbf{c}_i^\alpha)}{|\mathbf{c}_i^\alpha \cdot K^\alpha \mathbf{n}_i^\alpha|} k_i^\alpha\right] \left|\overline{M_1^\alpha M_2^\alpha}\right| \, ds_i \quad (i=1,2) \tag{2.1}$$

$$\mathbf{c}_i^\alpha := \overline{M_1^\alpha M_2^\alpha} \times \mathbf{f}_i'(s_i^\alpha), \quad \mathbf{n}_i^\alpha := \mathbf{n}(u_i^\alpha, \upsilon_i^\alpha)$$

where $\mathbf{v}_i^\alpha$ is the principle normal unit vector and k_i^α is the curvature of the normal section of the surface $K^\alpha Q$ at the point M_i^α in the direction of $\mathbf{f}_i'(s_i^\alpha)$.

The proof is given in the Appendix (Section 7).

To find the value of k_i^α in (2.1) when the quantities u_i, υ_i, s_i $(i=1, 2)$ are known it is required to find $du_i/d\upsilon_i$ which can be done by dividing the first of Eqs (1.1) by the second equation for given i. It is then sufficient to use Euler's theorem on normal sections [5]. Knowing $du_i/d\upsilon_i$ we can also calculate $|\mathbf{c}_i^\alpha \cdot K^\alpha \mathbf{n}_i^\alpha|$ since

$$|\mathbf{c}_i^\alpha \cdot K^\alpha \mathbf{n}_i^\alpha| = |\{[\mathbf{g}(u_2^\alpha, \upsilon_2^\alpha) - \mathbf{g}(u_1^\alpha, \upsilon_1^\alpha)] \times \mathbf{h}_i^\alpha\} \cdot \mathbf{n}_i^\alpha|$$

where $\mathbf{h}_i^\alpha := (K^\alpha)^{-1}\mathbf{f}_i'(s_i^\alpha)$ is the unit vector, tangential to Q at the point N_i^α in the direction of $du_i/d\upsilon_i$.

3. THE CHARACTERISTIC POINTS

We shall specify the relative positions of Q, L_1 and L_2 possessing certain qualitative features.

Suppose $\mathbf{c}_i^{\alpha_0}=0$ for some i and α_0, i.e. $\mathbf{f}_i'(s_i^{\alpha_0}) \parallel \overline{M_1^{\alpha_0} M_2^{\alpha_0}}$. Then $ds_i/d\psi = 0$, i.e. when $d\psi$ is finite, this position is stationary for s_i. If this equality holds for only a single i, then it will be violated immediately when further rolling occurs, i.e. an instantaneous stop occurs for s_i. A change in the direction of motion of the point M_i^α after the instant α_0 is typical. If $\mathbf{c}_1^{\alpha_0} = \mathbf{c}_2^{\alpha_0} = 0$, the points M_1^α and M_2^α remain stationary when rotation occurs.

If $\mathbf{c}_i^{\alpha_0} \neq 0$, $\mathbf{c}_i^{\alpha_0} \cdot K^{\alpha_0}\mathbf{n}_i^{\alpha_0} = 0$ and $k_i^{\alpha_0} \neq 0$ a similar situation occurs. The only difference is that if these conditions are satisfied for $i=1$ and $i=2$, the stopping of the points M_1^α and M_2^α is, in general, instantaneous, with the exception of the case when $\mathbf{n}_1^{\alpha_0} \parallel \mathbf{n}_2^{\alpha_0} \parallel \overline{N_1^{\alpha_0} N_2^{\alpha_0}}$, the points M_1^α and M_2^α then remain stationary.

The opposite occurs when

$$\frac{\mathbf{c}_i^\alpha \cdot \mathbf{f}_i''(s_i^\alpha)}{|\mathbf{c}_i^\alpha|^2} = \frac{\operatorname{sgn}(\mathbf{v}_i^\alpha \cdot \mathbf{c}_i^\alpha)}{|\mathbf{c}_i^\alpha \cdot K^\alpha \mathbf{n}_i^\alpha|} k_i^\alpha \quad \text{when } \alpha = \alpha_0 \tag{3.1}$$

In this case $ds_i/d\psi = \infty$, i.e. when ds_i is finite we obtain $d\psi|_{\alpha=\alpha_0} = 0$. (Hence it follows that if equality (3.1) holds for only a single i, the relation $ds_j/ds_i = 0$ $(j \neq i)$ is satisfied when $\alpha = \alpha_0$.) In this case it is typical that $d\psi$ changes sign after s_i passes through $s_i^{\alpha_0}$, and therefore if (3.1) only holds for a single i, a change in the direction of motion of the point M_j^α $(j \neq i)$ is also typical when this passage occurs.

For equality (3.1) to be satisfied it is sufficient that, in particular, the line L_i should make a contact with the surface $K^{\alpha_0}Q$ at the point $M_i^{\alpha_0}$ of higher than the second order. But such contact is not necessary which can be proved by putting, for instance

$$K^{\alpha_0}Q = \{(x,y,z):\ z = x^2 - y^2\},\quad M_1^{\alpha_0}(0,0,0),\quad L_1 = \{(x,y,0):\ (x+y)^2 = x - y\},\quad M_2^{\alpha_0}(0,1,0).$$

The surface $K^\alpha Q$ was assumed above to have only a single common point M_i^α with each of the lines L_i. If this requirement is not imposed, two versions of the formulation of the problem are possible. Thus, we can assume that the surface $K^\alpha Q$ is permeable for the lines L_i. This means that $K^\alpha Q \cap L_i$ can also include other points, besides M_i^α, for which there is no contact requirement. In this case the use of system (1.1) is the same as when $K^\alpha Q \cap L_i = \{M_i^\alpha\}$.

In the second form of the statement of the problem, which is more natural in the context of mechanics (the rolling of a rigid body of fixed shape), the lines L_i should not be inside $K^\alpha Q$. Then, during rolling characteristic points of yet another type can occur, specified by the values $\alpha = \alpha_0$ for which the surface $K^{\alpha_0}Q$ has more than one common point with at least one of the lines L_i. Here, when the rotation continues, the appearance of discontinuities of the first kind of the functions $\alpha \mapsto u_i^\alpha$, v_i^α, s_i^α when $\alpha = \alpha_0$ is typical. An example will be given in Section 5.

4. THE CHOICE OF THE ROLLING PARAMETER

It is natural to introduce the variable $\psi = \int |d\psi|$ by summing small angles of rotation of the surface $K^\alpha Q$ from the initial position to any current position. This variable can be used in the system of equations (1.1) by putting $\alpha = \psi$, which preserves its symmetrical form. To fix the geometrical meaning of the argument we must add Eq. (2.1). (This is sufficient for any single i, since if $d\psi$ is eliminated from Eqs (2.1) with $i = 1, 2$ we obtain the consequence from system (1.1)). The drawback of introducing ψ is the fact that, in the general case, it has no direct geometrical meaning, if the vector $\overline{M_1^\alpha M_2^\alpha}$ changes direction, because an increment of ψ depends not only on the initial and final positions of $K^\alpha Q$ but also on the whole method of transferring from one position to another.

When investigating the dynamics of rolling it is natural to assume that α in Eqs (1.1) is the time t. In this case we must add to Eqs (1.1) the equation which specifies the velocity of motion of the representative point along a trajectory in the phase space. Suppose the motion of the body, bounded by the surface $K^t Q$, occurs in a potential force field, and suppose the potential energy of the body, when the points (u_1, v_1), $(u_2, v_2) \in Q$ coincide with the points s_1 and s_2 of the lines L_1 and L_2 respectively, is equal to $U(u_1, v_1, u_2, v_2, s_1, s_2)$ and the moment of inertia of this body about the axis passing through the points (u_1, v_1) and (u_2, v_2) is equal to $I(u_1, v_1, u_2, v_2)$. The necessary supplementary equation, if energy dissipation is ignored, then has the form

$$\frac{1}{2} I(u_1, v_1, u_2, v_2)\left(\frac{d\psi}{dt}\right)^2 + U(u_1, v_1, u_2, v_2, s_1, s_2) = \text{const}$$

where the expression for $d\psi$ must be substituted from (2.1).

5. EXAMPLE 1. ROLLING OF A CIRCULAR CYLINDER ALONG PARALLEL CATENARIES

Consider the simplest kind of rolling of a circular cylinder of radius R along the pair of parallel catenaries. Here we have

$$Q\colon\ \mathbf{r} = R\cos v\mathbf{i} + u\mathbf{j} + R\sin v\mathbf{k}$$

$$L_i\colon\ \mathbf{r} = a\,\text{arsh}(s/a)\mathbf{i} + (-1)^i l\mathbf{j} + (a^2 + s^2)^{1/2}\mathbf{k} \quad (i = 1, 2)$$

where $a, l > 0$ are parameters. We shall assume that during rolling the generatrices of the cylinder remain parallel to the y-axis. All possible motions are then obtained from a single motion by simple translations and rotations. We therefore put

$$u_i \equiv (-1)^i l \quad (i = 1, 2), \quad v_1 \equiv v_2 =: v, \quad s_1 \equiv s_2 =: s$$

System (1.1) reduces to the single equation $|Rv'| = |s'|$, i.e. we can assume that $Rv \equiv s + \text{const}$. Furthermore, formula (2.1) gives

$$d\psi = \left(\frac{1}{R} - \frac{a}{s^2 + a^2}\right) ds$$

Hence, we obtain that if $R < a$, there are no characteristic points mentioned in Section 3 and we can put

$$\upsilon = \frac{s}{R}, \quad \psi = \frac{s}{R} - \operatorname{arctg}\frac{s}{a}$$

When $R = a$ a stationary point $s = 0$ for $\psi(s)$ appears.

Now suppose that $R > a$; this case is of greater interest. If the surface $K^\alpha Q$ is assumed to be permeable for the lines L_i, it can roll uninterruptedly along them, i.e. s can be taken as the parameter α. However, when s increases in the interval $(-[a(R-a)]^{1/2}, [a(R-a)]^{1/2})$, we then have $d\psi < 0$, i.e. rotation occurs in the negative direction.

If the surface $K^\alpha Q$ is not permeable for the lines L_i when $R > a$, the picture is different. When $s = \pm a\lambda(R/a)$ the surface $K^\alpha Q$ has two points of contact with each of the lines L_i, where $\lambda(p)$ is the single positive root of the equation

$$\lambda = \operatorname{sh}[p\lambda(1+\lambda^2)^{-1/2}] \quad (p > 1)$$

Hence, two characteristic points of the latter type described in Section 3 appear here and s can only take the values in the intervals $(-\infty, -a\lambda(R/a))$ and $(a\lambda(R/a), \infty)$. When the surface $K^\alpha Q$ reaches these points it "passes" continuously from one interval to another so that s and υ undergo discontinuities of the first kind but φ remains continuous.

6. EXAMPLE 2. ROLLING OF A SPHERE ALONG A PAIR OF PLANE LINES WHICH ARE SYMMETRICAL WITH RESPECT TO EACH OTHER

The characteristic feature of the rolling of a sphere is that one more finite equation connecting s_1 with s_2 (in general, quite lengthy) appears here. As a result the set of geometrically different motions becomes two-dimensional, if the points of the sphere differ, and zero-dimensional (discrete), if they do not differ.

Consider the simple case when the lines L_i lie in the x-y plane and are symmetrical to one another about the x-axis, i.e.

$$\mathbf{f}_1(s) = x(s)\mathbf{i} + y(s)\mathbf{j}, \quad \mathbf{f}_2(s) = x(s)\mathbf{i} - y(s)\mathbf{j}$$

where $x(\cdot)$ and $y(\cdot)$ are specified sufficiently smooth functions, and

$$y(s) > 0, \quad x'(s) > 0, \quad [x'(s)]^2 + [y'(s)]^2 = 1$$

Let Q be a sphere of radius R and u and υ be spherical coordinates with poles on the y-axis, i.e.

$$\mathbf{g}(u,\upsilon) = (R\sin\upsilon\cos u)\mathbf{i} + (R\cos\upsilon)\mathbf{j} + (R\sin\upsilon\sin u)\mathbf{k}$$

From considerations of symmetry we obtain

$$s_1 \equiv s_2 =: s, \quad u_1 \equiv u_2 =: u, \quad \upsilon_1 \equiv \pi - \upsilon_2 =: \upsilon \in (0, \pi/2)$$

Equality (1.2) takes the form $R\cos\upsilon = y$ and defines the quantities $\upsilon(s)$, while the first equation of (1.1) takes the form

$$R^2(\sin^2 u\, du^2 + d\upsilon^2) = ds^2$$

whence we obtain the equation

$$du = \pm(R^2 x'^2 - y^2)^{1/2}(R^2 - y^2)^{-1} ds \tag{6.1}$$

for the function $s \mapsto u(s)$. Here the upper (lower) sign is chosen if the z-coordinate of the centre of the sphere $K^\alpha Q$ is greater (respectively less) than zero.

Formula (6.1) implies, in particular, the necessary condition for rolling to be possible

$$y(s) \leq Rx'(s) \tag{6.2}$$

When this inequality becomes an equality the centre of the sphere $K^\alpha Q$ lies in the x–y plane.

To apply formula (2.1) we find

$$\mathbf{c}_i^\alpha = 2x'y\mathbf{k}, \quad \mathbf{f}_i'(s_i) = x''\mathbf{i} - (-1)^i y''\mathbf{j}, \quad k_i^\alpha = 1/R$$

$$K^\alpha \mathbf{n}_i^\alpha = \frac{1}{R}\left[-\frac{yy'}{x'}\mathbf{i} - (-1)^i y\mathbf{j} \mp \frac{(R^2x'^2 - y^2)^{1/2}}{x'}\mathbf{k}\right]$$

with the same rule of signs as in formula (6.1). Hence we obtain

$$ds = \mp\{R^2[x'(s)]^2 - [y(s)]^2\}^{1/2} d\psi \tag{6.3}$$

Thus, the points, at which inequality (6.1) become an equality, are characteristic points, namely, stationary for s; ψ has no stationary points here and, hence, ψ can be globally taken as the parameter α. Note that (6.1) and (6.3) imply the inequality $|du/d\psi| \leq 1$ which becomes an equality only if $y'(s) = 0$.

From (6.3) we obtain the differential equation

$$d^2s/d\psi^2 + y(s)y'(s) - R^2x'(s)x''(s) = 0$$

showing, among other things, what happens near the characteristic values of s; namely, if $s = s^*$ is such a value and the condition

$$y'(s^*) \neq Rx''(s^*) \tag{6.4}$$

holds then for $s = s^*$ we obtain $ds/d\psi = 0$, $d^2s/d\psi^2 \neq 0$, i.e. when ψ passes through $\psi^*(s|_{\psi=\psi^*} = s^*)$ the function $\psi \mapsto s(\psi)$ changes the direction of its variation. This means that the centre of the sphere $K^\psi Q$ crosses the x–y plane and the sphere starts to move backwards. If inequality (6.2) is strictly satisfied inside a certain interval $s^{**} \leq s \leq s^*$ and becomes an equality at the ends of the interval, and condition (6.4) is satisfied at both ends, then, when ψ increases continuously, the sphere moves periodically with a period (over ψ) equal to

$$\Psi = 2\int_{s^{**}}^{s^*} \{R^2[x'(s)]^2 - [y(s)]^2\}^{-1/2} ds$$

Suppose, for example, that the lines L_i have the equation $y = (-1)^{i-1} a\,\mathrm{ch}(x/a)$ with $0 < a < R$ and suppose s is measured from the point $(0, a)$. The critical points are then $s = \pm[a(R-a)]^{1/2}$ and the period (over ψ) is equal to

$$\Psi = 4\int_0^{[a(R-a)]^{1/2}} (a^2+s^2)^{1/2}[a^2R^2 - (a^2+s^2)^2]^{-1/2} ds = \frac{4a}{[R(R-a)]^{1/2}}\Pi\left(\frac{\pi}{2}, \frac{R-a}{R}\left(\frac{R-a}{R+a}\right)^{1/2}\right)$$

where Π is the elliptic integral of the third kind.

If we require that $y'(s^*) = Rx''(s^*)$ instead of condition (6.4), then in a sufficient vicinity of s^* the inequality

$$|R^2[x'(s)]^2 - [y(s)]^2| \leq C^2(s-s^*)^2 \quad (C = \text{const} > 0)$$

holds and it therefore follows from (6.3) that $|ds/d\psi| \leq C|s-s^*|$. Hence, when $s \neq s^*$ we have

$$\frac{d}{d\psi}(e^{C\psi}|s-s^*|) = Ce^{C\psi}|s-s^*| + e^{C\psi}\operatorname{sgn}(s-s^*)\frac{ds}{d\psi} \geq e^{C\psi}\left(C|s-s^*| - \left|\frac{ds}{d\psi}\right|\right) \geq 0$$

Hence $e^{C\psi}|s-s^*| > \text{const} > 0$ and, therefore, $s \to s^*$ only when $\psi \to \infty$.

Thus, the sphere $K^\nu Q$ approaches the characteristic point only asymptotically, making an infinite number of revolutions. Since in the limiting position, when $s = s^*$, rotation without changing s is possible, we obtain that the case under consideration is realized if and only if $y(s^*) = R$, $y'(s^*) = 0$.

7. APPENDIX. PROOF OF THE THEOREM

We will first prove the lemma on the change in curvature of a line when projected. We shall consider orthogonal projection onto a plane P' and denote the images after projection by primes. Let M be a point of a sufficiently smooth line L. We will establish a relation between the curvature k of the line L at the point M and the curvature k' of the corresponding line L' at the point M'.

Lemma. The following formula holds:

$$k'\cos^3\beta = k\cos\gamma$$

where $\beta \in [0, \pi/2]$ is the angle of inclination of the line L to the plane P' at the point M and $\gamma \in [0, \pi/2]$ is the angle between P' and the touching plane P of the line L at the point M.

Proof. Without loss of generality, we will assume that the line L lies in the plane P. Let l be the tangent to L at the point M, τ be the unit vector of this tangent, $\mathbf{n}$ be the unit vector of the principle normal to L at M, and $\mathbf{d} \perp P'$ be the unit vector. Choosing $A \in l$, $B \in L$, $\overline{AB} \parallel \mathbf{n}$ we have $|\overline{AB}| \sim k|\overline{MB}|^2/2$ as $A \to M$.

Furthermore, projecting onto P', we obtain

$$\overline{M'A'} = \overline{MA} - (\overline{MA}\cdot\mathbf{d})\mathbf{d}, \quad \overline{A'B''} = \overline{AB} - (\overline{AB}\cdot\mathbf{d})\mathbf{d}$$

Therefore, the modulus p of the projection of the vector $\overline{A'B'}$ onto the principle normal to L' at M' is equal to

$$|\overline{A'B'}|\left[1 - \left(\frac{\overline{M'A'}\cdot\overline{A'B'}}{|\overline{M'A'}||\overline{A'B'}|}\right)^2\right] = \{|\overline{AB}|[1-(\mathbf{n}\cdot\mathbf{d})^2]^{1/2}\}\left\{1 - \frac{(\boldsymbol{\tau}\cdot\mathbf{d})^2(\mathbf{n}\cdot\mathbf{d})^2}{[1-(\boldsymbol{\tau}\cdot\mathbf{d})^2][1-(\mathbf{n}\cdot\mathbf{d})^2]}\right\}^{1/2} =$$

$$= |\overline{AB}|\frac{[1-(\boldsymbol{\tau}\cdot\mathbf{d})^2 - (\mathbf{n}\cdot\mathbf{d})^2]^{1/2}}{[1-(\boldsymbol{\tau}\cdot\mathbf{d})^2]^{1/2}} = |\overline{AB}|\frac{\cos\gamma}{\cos\beta}$$

However, it can be easily verified that $k' = 2\lim(p|\overline{M'A'}|^{-1})$ as $A \to M$. Hence

$$k' = 2\lim_{A\to M}\left\{|\overline{AB}|\frac{\cos\gamma}{\cos\beta}|\overline{MA}|^{-2}[1-(\boldsymbol{\tau}\cdot\mathbf{d})^2]^{-1}\right\} = 2\frac{\cos\gamma}{\cos^3\beta}\lim_{A\to M}\frac{|\overline{AB}|}{|\overline{MA}|^2} = k\frac{\cos\gamma}{\cos^3\beta}$$

This proves the lemma.

Considering now the proof of the theorem, we will first examine the special case when $K^\alpha Q$ is a cylindrical surface with generatrices parallel to the $M_1^\alpha M_2^\alpha$ axis, and the line L_i lies in the plane $P_i^\alpha \perp \overline{M_1^\alpha M_2^\alpha}$. We also assume that the vectors $\mathbf{f}_i'(s_i^\alpha)$ and ν_i^α are directed opposite to the vector c_i^α and that $k_{L_i}^\alpha := |\mathbf{f}_i'(s_i^\alpha)| < k_i^\alpha$. We replace the lines L_i and $K^\alpha Q \cap P_i^\alpha$ by their osculating circles at the point M_i^α and we denote the centre of the second circle by O_i^α. Then, after rotation we obtain, omitting small higher-order terms

$$d\psi = \angle O_i^\alpha M_i^\alpha O_i^{\alpha+d\alpha} = k_i^\alpha |\overline{O_i^\alpha O_i^{\alpha+d\alpha}}| = k_i^\alpha (1 - k_{L_i}^\alpha / k_i^\alpha) |\overline{M_i^\alpha M_i^{\alpha+d\alpha}}| = (k_i^\alpha - k_{L_i}^\alpha) ds_i$$

Consideration of the remaining forms of arrangement of the vectors $\mathbf{f}''(s_i^\alpha)$ and $\boldsymbol{\nu}_i^\alpha$ relative to c_i^α shows that, in all these versions, in the special case investigated, the equality

$$d\psi = \{k_{L_i}^\alpha \operatorname{sgn}[\mathbf{f}_i''(s_i^\alpha) \cdot \mathbf{c}_i^\alpha] - k_i^\alpha \operatorname{sgn}(\boldsymbol{\nu}_i^\alpha \cdot \mathbf{c}_i^\alpha)\} ds_i \tag{7.1}$$

holds for any relation between $k_{L_i}^\alpha$ and k_i^α.

Now let the surface $K^\alpha Q$ and the lines L_i be arbitrary. Up to third-order infinitesimal terms in ds_i, replace the line L_i by its osculating circle $\overline{L}_i^\alpha$ at the point M_i^α and replace the track of this line on the surface $K^\alpha Q$, which is produced when the latter rolls, by the line D_i^α, of the intersection of $K^\alpha Q$ with the plane passing through M_i^α parallel to the vectors $\mathbf{f}_i'(s_i^\alpha)$ and c_i^α. If we now draw a cylindrical surface through D_i^α with generatrices parallel to the vector $\overline{M_1^\alpha M_2^\alpha}$, and project $\overline{L}_i^\alpha$ and D_i^α onto the plane $S \perp \overline{M_1^\alpha M_2^\alpha}$, then the general case under consideration reduces to the special case analysed above.

Putting $\boldsymbol{\tau}_i^\alpha := \mathbf{f}_i'(s_i^\alpha)$, $\mathbf{f}_{12}^\alpha := M_1^\alpha M_2^\alpha / |M_1^\alpha M_2^\alpha|$ for brevity, we find the curvature $\bar{k}_i^\alpha$ of the line D_i^α at the point M_i^α

$$\bar{k}_i^\alpha = \frac{k_i^\alpha}{|\cos(K^\alpha \mathbf{n}_i^\alpha, \boldsymbol{\tau}_i^\alpha \times \mathbf{f}_{12}^\alpha)|} = k_i^\alpha \frac{|\boldsymbol{\tau}_i^\alpha \times \mathbf{f}_{12}^\alpha|}{|(\boldsymbol{\tau}_i^\alpha \times \mathbf{f}_{12}^\alpha) \cdot K^\alpha \mathbf{n}_i^\alpha|}$$

by Meusnier's theorem [5].

We now use the lemma. Since, in this case, β is the complementary angle between $\boldsymbol{\tau}_i^\alpha$ and $\mathbf{f}_{12}^\alpha$ and γ is the angle between the vectors $\boldsymbol{\tau}_i^\alpha \times (\boldsymbol{\tau}_i^\alpha \times \mathbf{f}_{12}^\alpha) = (\boldsymbol{\tau}_i^\alpha \cdot \mathbf{f}_{12}^\alpha)\boldsymbol{\tau}_i^\alpha - \mathbf{f}_{12}^\alpha$ and $\mathbf{f}_{12}^\alpha$ then, by the lemma, we find the curvature after projection

$$\bar{k}_i^{\alpha\prime} = \bar{k}_i^\alpha \frac{|\cos[(\boldsymbol{\tau}_i^\alpha \cdot \mathbf{f}_{12}^\alpha)\boldsymbol{\tau}_i^\alpha - \mathbf{f}_{12}^\alpha, \mathbf{f}_{12}^\alpha]|}{\sin^3(\boldsymbol{\tau}_i^\alpha, \mathbf{f}_{12}^\alpha)} = \frac{\bar{k}_i^\alpha}{1 - (\boldsymbol{\tau}_i^\alpha \cdot \mathbf{f}_{12}^\alpha)^2} =$$

$$= k_i^\alpha |(\boldsymbol{\tau}_i^\alpha \times \mathbf{f}_{12}^\alpha) \cdot K^\alpha \mathbf{n}_i^\alpha|^{-1} [1 - (\boldsymbol{\tau}_i^\alpha \cdot \mathbf{f}_{12}^\alpha)^2]^{-1/2}$$

The curvature $k_{L_i}^{\alpha\prime}$ of the projected line L_i is found in a similar way. Here β is the same as above and γ is the angle between $\mathbf{f}_i''(s_i^\alpha) \times \boldsymbol{\tau}_i^\alpha$ and $\mathbf{f}_{12}^\alpha$. (We assume that $\mathbf{f}''(s_i^\alpha) \neq 0$, otherwise we have $k_{L_i}^{\alpha\prime} = 0$.). Putting $\mathbf{p}_i^\alpha = \mathbf{f}_i''(s_i^\alpha) / |f_i''(s_i^\alpha)|$ we obtain by the lemma

$$k_{L_i}^{\alpha\prime} = |\mathbf{f}_i''(s_i^\alpha)| \cdot |(\mathbf{p}_i^\alpha \times \boldsymbol{\tau}_i^\alpha) \cdot \mathbf{f}_{12}^\alpha| [1 - (\boldsymbol{\tau}_i^\alpha \cdot \mathbf{f}_{12}^\alpha)^2]^{-3/2}$$

Finally, note that the element of ds_i of the length of the arc of the line L_i on projection becomes shorter by the formula

$$ds_i' = [(1 - \boldsymbol{\tau}_i^\alpha \cdot \mathbf{f}_{12}^\alpha)^2]^{1/2} ds_i$$

Hence, applying formula (7.1) to ds_i' and reverting to the initial notation, after simple reduction we obtain (2.1).

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